

# Generalized Steiner's Problem and its Solution with the Concepts in Field Theory

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## Abstract

We generalized the Steiner's shortest line problem and found its connection with the concepts in classical field theory. We solved the generalized Steiner's problem by introducing a conservative potential and a dissipative force in the field and gave a computing method by using a testing point and a corresponding iterative curve.

## 1 Steiner's Problem

In the  $\Re^2$  Euclidean space, there are  $n$  points  $\alpha_i (i = 1, \dots, n)$ . Find a point  $\alpha_{steiner}$  that the sum of its distances to the  $n$  points is a minimum:

$$\forall \alpha \in \Re^2, \sum_{i=1}^n |\alpha_{steiner} - \alpha_i| \leq \sum_{i=1}^n |\alpha - \alpha_i|. \quad (1)$$

This is the original Steiner's problem[1].

## 2 Generalized Steiner's Problem

We would like to expand this problem to a more general situation:

In the  $D$ -dimension differentiable linear space  $E^D$ , there are  $n$  points  $\alpha_i (i = 1, \dots, n)$ . How to find a point  $\alpha_{steiner}$  that the sum of its defined distances  $dis$  to the  $n$  points is a minimum:

$$\forall \alpha \in E^D, \sum_{i=1}^n dis(\alpha_{steiner} - \alpha_i) \leq \sum_{i=1}^n dis(\alpha - \alpha_i). \quad (2)$$

We call the point  $\alpha_{steiner}$  as the Steiner point.

## 3 Mapping Concepts onto Field Theory

If we map the concept of 'distance' in the generalized Steiner's problem onto the concept of 'force potential' in the Filed Theory, we will obtain another equivalent form of the generalized problem. Let's consider that a particle at the testing point has conservative

force interactions with each of the  $n$  points and therefore has potentials  $U_i(\alpha - \alpha_i)$  ( $i = 1, \dots, n$ ). The Steiner point makes the sum of the  $n$  potentials to be a minimum. If we let

$$U_i(\alpha - \alpha_i) = \text{dis}(\alpha - \alpha_i), (i = 1, \dots, n) \quad (3)$$

and find the Steiner point in this conservative field, the generalized Steiner's problem is solved.

Obviously, the Steiner point should be in the set:

$$\alpha_{steiner} \in \text{set} \left\langle \alpha \mid \nabla \left( \sum_{i=1}^n U_i(\alpha - \alpha_i) \right) = 0 \right\rangle. \quad (4)$$

Thus  $\alpha_{steiner}$  can be identified by comparing each elements in the set.

## 4 Testing and Iterating Methods

The next question is 'How to get that set by a computing method?' We would like to use a testing point  $\alpha_{testing}$  and the interative curve  $L_{iterative}$  and introduce a conservative potential  $U(\alpha) = \sum_{i=1}^n U_i(\alpha - \alpha_i)$  and a dissipative force. Note that the dissipative force acts only when a particle moves. If  $\alpha_{testing}$  is not in  $\text{set} \langle \alpha \mid \nabla \left( \sum_{i=1}^n U_i(\alpha - \alpha_i) \right) = 0 \rangle$ , we put a point-like large-mass rest particle at  $\alpha_{testing}$ . The conservative force exerted to the particle is[2]:

$$- \nabla U(\alpha_{testing}). \quad (5)$$

The particle is intended to move along the direction of the conservative force, but the dissipative force is strong enough to keep the particle moving in a quasi-static matter until rest at a point  $\alpha_s$ . ( $\alpha_s \in \text{set} \langle \alpha \mid \nabla \left( \sum_{i=1}^n U_i(\alpha - \alpha_i) \right) = 0 \rangle$ .)

If the particle moves slowly enough, its path from  $\alpha_{testing}$  to  $\alpha_s$  is regarded as an iterative curve, with which we can find  $\alpha_s$  by iterating points along the curve from  $\alpha_{testing}$ .

## 5 Iterative Curve and Conservative Potential

The next question is 'How to get the iterative curve  $L$  after the testing point is known?'. The iterative curve's tangent is everywhere parallel to the conservative force (For the D-dimemsion situation, the coordinates are  $Z_i$ , ( $i = 1, 2, 3, \dots, D$ ) and  $\lambda$  is a nonzero real number):

$$\left( 1, \frac{\partial Z_2}{\partial Z_1} \Big|_L, \frac{\partial Z_3}{\partial Z_1} \Big|_L, \dots, \frac{\partial Z_D}{\partial Z_1} \Big|_L \right) = \lambda \left( \frac{\partial U}{\partial Z_1}, \frac{\partial U}{\partial Z_2}, \frac{\partial U}{\partial Z_3}, \dots, \frac{\partial U}{\partial Z_D} \right). \quad (6)$$

Thus we get:

$$\frac{\partial Z_i}{\partial Z_1} \Big|_L = \frac{\partial U}{\partial Z_i} / \frac{\partial U}{\partial Z_1}, (i = 2, 3, \dots, D). \quad (7)$$

If the testing point  $\alpha_{testing} = (Z_1^{testing}, Z_2^{testing}, \dots, Z_D^{testing})$ , the iterative curve is governed by the following D-1 functions:

$$Z_i - \int_{Z_1^{testing}}^{Z_1} \left( \frac{\partial U}{\partial Z_i} / \frac{\partial U}{\partial Z_1} \right) dZ_1 = 0, (i = 2, 3, \dots, D). \quad (8)$$

## References

- [1] E. N. Gilbert, H. O. Pollak, *SIAM J. Appl. Math.*, **16** (1968)1-29
- [2] L. D. Landau, E. M. Lifshitz, *The Classical Theory of Fields*, Butterworth-Heinemann Press, Britain(1998)